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Estimation of Distributed Parameter Systems

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The estimation and control of large flexible space structures pose significant new control technology problems. One major problem area is the closed-loop stability of a structure that has been modeled with truncated modal dynamics. Stability problems arise due to the control design process beginning with a deficient model. This work provides the necessary conditions for the optimal estimation of infinite dimensional (partial-differential equation) systems. This approach can be particularly useful for initial control studies and for gaining considerable insight into what the optimal estimator for a truly infinite dimensional system should be. A detailed example of the estimation of the continuous shape of a string in tension is presented.

Nomenclature

a,b = arbitrary functions
B = boundary condition differential operator
D = differential operator
F = state dynamics differential operator
H = measurement differential operator
J = scalar performance index
K = estimator gains differential operator
P = Ricatti differential operator
Q = spectral density of process disturbance
R = spectral density of measurement noise
t = time variable
v = measurement noise
w = process noise
x,ξ = spatial variable
y = spatially continuous state variable measurement
z = measurement
Γ = process disturbance distribution matrix
Γ = boundary of Ω
λ = adjoint variable
Ω = spatial domain
δ = variation
$(,)$ = inner product
$\| \|$ = norm
$( )^T$ = transpose
$( )^{-1}$ = inverse
$( )^*$ = adjoint
$\hat{( )}$ = estimate

Introduction

As spacecraft become larger and more flexible, the equations of motion needed to accurately model the dynamic behavior of these spacecraft become more complex. Higher order finite-element (FE) models must be synthesized by the structural analysts in response to spacecraft tending away from interconnected lumped mass configurations and toward continuously distributed configuations. More and more structural modes are passed on to the control analysts as control bandwidths encompass an ever increasing number of modes. Consequently, the problem of onboard state estimation for the purposes of attitude control, shape control, stationkeeping, and appendage articulation can become unwieldy. Furthermore, serious well-known stability problems may arise due to poorly modeled systems, truncated modal models, time-varying system parameters, etc.[1]

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One possible method of circumventing the problems resulting from the increased number of modes to be controlled is by avoiding modal models. Working directly with partial-differential equation (PDE) models of continuous spacecraft is a viable alternative for preliminary control designs. In fact, some structures such as solar panels, large antennas, and astromasts may be more easily modeled with PDE than with FE, particularly when the inclusion of structural analysts is the control design loop results in excessive time delays.

The motivation behind the "local" approach to be discussed here is that the PDE for a continuous spacecraft describes the acceleration of each physical point on the spacecraft in terms of differential operators; that is, in terms of the behavior of the spacecraft within a local neighborhood of each point. Since sensors measure, and disturbances affect, local variables, a reasonable job of state estimation may be accomplished with local state estimation. The entire design procedure would avoid entirely the problems associated with high-order (possibly truncated) modal modes.

The control analog of this problem has already been addressed in some detail.[2] It has been shown that an optimal control law for a free-free flexible beam very closely resembles a local controller. This means that if a modal controller for this system is designed, the feedback law may actually be represented more simply if the control law is expressed in physical coordinates.

Breakwell[3] has also obtained some results for the optimal control of a continuous flexible beam using PDE in which a symmetric root locus approach is used to determine the optimal closed-loop root positions. Following this step, the continuous control gains needed to move the roots to their optimal closed-loop locations must be determined.

The purpose of this paper is to present a PDE estimation and control problem formulation that contains sufficient generality to encompass a wide variety of continuous control system design problems within a single analytical framework. By drawing on a simplified example, a string in tension, some insight to the control of continuous structures is obtained, and some generalizations for their control can be made. Although the estimation problem is discussed here, the results can be easily extended to include the control problem also. A procedure for the direct determination of the continuous optimal feedback gain is given, following the format used in the control of systems governed by ordinary differential equations. It is hoped in advance to find if and when the retention of physical coordinates will produce an estimator that can be characterized only in terms of local state feedback, and avoid the problems associated with modal truncation.

Analysis

Consider a general partial-differential equation of motion in state variable format. This equation represents the dynamics of the spatially continuous state vector in some spatial domain, Ω, with boundary conditions prescribed on the boundary, Γ, of this domain. The initial state of the system is given. It is desired to obtain an estimate of the state of the system as time evolves, from noisy measurements in the presence of disturbances. The mathematical model is given as

$$\dot{y}(x,t) = F(D)y(x,t) + \Gamma(x,t)w(x,t)$$

$$y(x,t_0) \quad \text{and} \quad B(D)y(x,t)\,|_{\Gamma} \text{ given}$$

$$z(x,t) = H(D)y(x,t) + v(x,t) \tag{1}$$

$F(D)$, $H(D)$, and $B(D)$ are linear differential operators in the spatial variable. By allowing H to be an operator, position, slope, curvature, and other measurements may all be treated with equal facility. Were it not for the presence of the differential operators in Eq. (1), the mathematical formulation would be identical to the ordinary differential equation state variable format. [It will therefore be notationally convenient to omit future appearances of all superfluous symbols. It should be remembered that the model actually analyzed is written in all of its detail only in Eq. (1).] In order to obtain a state estimate from the measurements, a performance criteria is selected which weighs the relative importance of the process and measurement disturbance. Minimization of the expected value of this performance index is one way of obtaining the optimal state estimate.

$$J = \frac{1}{2}\int_{t_0}^{t_f}\int_{\Omega} (\hat{w}^T Q^{-1}\hat{w} + \hat{v}^T R^{-1}\hat{v})\,\mathrm{d}x\mathrm{d}t \tag{2}$$

Adjoining the constraint dynamics in Eq. (1) to the performance index in Eq. (2) gives

$$J = \int_{t_0}^{t_f}\int_{\Omega} \frac{\{\hat{w}^T Q^{-1}\hat{w} + (z-Hy)^T R^{-1}(z-Hy)}{2}$$

$$+\lambda^T(-\dot{y} + Fy + \Gamma\hat{w})\}\,\mathrm{d}x\mathrm{d}t \tag{3}$$

The necessary condition for the minimization of the performance index is that the first variation of J vanishes for arbitrary admissible variations. This variation yields

$$0 = \delta J = \int_{t_0}^{t_f}\int_{\Omega} (\hat{w}^T Q^{-1} + \lambda^T\Gamma)\delta\hat{w}\,\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_0}^{t_f}\int_{\Omega} \{-\delta y^T H^T R^{-1}(z-Hy) - \lambda^T\delta\dot{y} + \lambda^T F\delta y\}\,\mathrm{d}x\mathrm{d}t \tag{4}$$

Since δw in Eq. (4) is arbitrary, the optimality condition is obtained.

$$\hat{w} = -Q\Gamma^T\lambda \tag{5}$$

Two steps are now required to isolate the δy multiplier in Eq. (4). The first is to integrate the $\lambda^T\delta\dot{y}$ term by parts to obtain the final conditions on the adjoint variable, $\lambda^T(x,t_f) = 0$, and also an integrated term, $-\dot{\lambda}^T\delta y$. The second step requires the isolation of the δy terms from their *operators* F and H^T. A short digression is required.

Let $(\cdot,\cdot)$ define an inner product. If T is some operator, then its adjoint operator T^* is defined as

$$(a,Tb) = (T^*a,b)$$

Returning now to the original problem at hand, define the inner product for this analysis to be

$$(a,b) = \int_{\Omega} a(x)b(x)\mathrm{d}x$$

The δy terms in Eq. (4) must now be isolated from the H and F operators. As usual, δy is arbitrary, therefore, a further necessary condition for the minimizing of the performance index is that the resulting terms multiplying δy sum to zero. This yields the partial-differential equation for the adjoint variable, λ, and its boundary condition. Using the star notation to denote adjoint operators, this may be written as

$$\dot{\lambda} + F^*\lambda - H^{*T}R^{-1}(z-Hy) = 0 \qquad B^*\lambda = 0 \tag{6}$$

The closed-loop dynamics of the optimal estimator and the adjoint variable is now written as

$$\begin{bmatrix}\dot{y}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix} F & -\Gamma Q\Gamma^T \\ -H^{T*}R^{-1}H & -F^* \end{bmatrix}\begin{bmatrix} y\\ \lambda\end{bmatrix} + \begin{bmatrix} 0 \\ H^{T*}R^{-1}H\end{bmatrix} z$$

$$y(t_0) \text{ given} \quad \lambda(t_f) = 0 \quad By = 0 \quad B^*\lambda = 0 \tag{7}$$

As usual, it is desirable to write the estimator alone in the form

$$\dot{\hat{y}} = F\hat{y} + K(z - H\hat{y}) \tag{8}$$

where K may be a differential operator estimator gain.

Equation (7) is a linear inhomogeneous two-point boundary value problem. As such, a solution of the form

$$y = \hat{y} - P\lambda$$

where P is a linear-differential operator, may be assumed. Making this substitution into Eq. (7) yields

$$\dot{\hat{y}} = F\hat{y} + K(z - H\hat{y}) \qquad \hat{y}(t_0) = y(t_0)$$

$$\dot{P} = PF^* + FP + \Gamma Q\Gamma^T - PH^{*T}R^{-1}HP$$

$$P(0) = 0 \qquad K = PH^{T*}R^{-1} \tag{9}$$

where products of operators, of course, represent composite operations.

The problem of determining the optimal estimator gains for a spatially continuous system has now been reduced to a familiar format. The only differences are that the estimator gains are now in operator form and that the Ricatti equation for the estimator gain is in terms of a differential operator, P. That such a solution exists and is unique is not pursued here.[4] Rather, a detailed example problem is presented in the following section.

Flexible String Example

Consider the dynamics of the string in tension shown in Fig. 1. The partial-differential equation of motion is given by

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + w$$

$$y(0,t) = y(1,t) = 0 \qquad y(x,0) = 0 \tag{10}$$

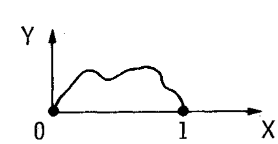

Fig. 1 The flexible string.

Assume a continuous position measurement, z, is available, and that it is desired to find the steady-state estimator that will minimize the expected value of the following performance index.

$$J=\frac{1}{2}\int_0^\infty\int_0^1 (\hat{w}^T Q^{-1}\hat{w}+\hat{v}^T R^{-1}\hat{v})\mathrm{d}x\mathrm{d}t \tag{11}$$

From Eqs. (10) and (11), the equivalent state variable formulation is obtained as

$$F=\begin{bmatrix} 0 & 1 \\ D^2 & 0 \end{bmatrix} \quad \Gamma=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H=[1,0] \tag{12}$$

Letting

$$(a,b)=\int_0^\infty\int_0^1 ab\mathrm{d}x\mathrm{d}t$$

the adjoint operator to F is given by[5]

$$F^*=\begin{bmatrix} 0 & D^2 \\ 1 & 0 \end{bmatrix} \tag{13}$$

Using the results of Eq. (9) and letting

$$P=\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

it can be shown that the steady-state optimal estimator can be obtained from the solution of

$$P_3+P_2-P_1R^{-1}P_1=0$$
$$P_4+P_1D^2-P_1R^{-1}P_2=0$$
$$D^2P_1+P_4-P_3R^{-1}P_1=0$$
$$D^2P_2+P_3D^2+Q-P_3R^{-1}P_2=0 \tag{14}$$

Again, it is pointed out that the products above denote composite operations. Even so, the feedback gains can be written symbolically as

$$\begin{bmatrix} K_y \\ K_v \end{bmatrix}=\begin{bmatrix} P_1/R \\ P_3/R \end{bmatrix}=\begin{bmatrix} \sqrt{2(D^2+(D^4+Q/R)^{1/2}} \\ D^2+(D^4+Q/R)^{1/2} \end{bmatrix} \tag{15}$$

where the square root of an operator is to be interpreted as the operator whose composite operation with itself yields the desired operator. In each case, results obtained using the above analysis will be compared with numerical results using a discretized 20-state representation of the string dynamics, with the program OPTSYS.[6]

In the following examples, the estimator gains determined using the differential operator approach will depend on the relative norms of D^4 and Q/R in Eq. (15). Obviously the norm of D^4 defined as

$$\|D^4\|^2=(y,D^4y)/(y,y)=(D^2y,D^2y)/(y,y)$$

can be made arbitrarily large. At this time, some notion of the modal concept does prove to be useful, although it is still not required. In general, higher-frequency modes in the time domain result in higher-frequency mode shapes in the spatial domain. The norm of D^4 on these eigenfunctions will then serve as a basis for which modes are low frequency, and which are high frequency. What will eventually be seen is that there will be distinct estimation approaches for the high-frequency modes vs the low-frequency modes.

Case I: $Q=10^4$, $R=1$

For Q/R large, or alternatively, for estimation of the component motion due to low-frequency behavior, an expansion of Eq. (15) gives

$$K_v=(Q/R)^{1/2}\left[1+\frac{D^2}{(Q/R)^{1/2}}+\frac{D^4}{2Q/R}\cdots\right]$$

$$K_y=2^{1/2}(Q/R)^{1/4}\left[1+\frac{D^2}{2(Q/R)^{1/2}}\cdots\right] \tag{16}$$

Regardless of the dynamic system involved, for Q/R large, the estimator gains as a function of position are approximately constants. Since they are not "operators," they will involve feedback of only local state information. In fact, Q/R large is precisely the condition needed to guarantee the optimality of local estimation (or local control in the control problem). The estimator for this problem will then be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y \\ \dot{y} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ D^2 & 0 \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix}+\begin{bmatrix} \sqrt{2}\,(Q/R)^{1/4} \\ (Q/R)^{1/2} \end{bmatrix}(z-\hat{y}) \tag{17}$$

Notice that the estimate error tends to zero for all modes, not just for the low-frequency modes. That is to say, truncation of the differential operator to include only the leading term does not result in any instabilities.

In order to graphically display the estimator gains, two spatial variables, x and ξ, are required. The feedback gain, $K(x,\xi)$, will then be understood to mean the feedback gain associated with the sensor at location ξ to update the state at location x. Local control is then easily written as

$$K(x,\xi)\neq 0 \quad \text{if} \quad x=\xi$$
$$K(x,\xi)=0 \quad \text{if} \quad x\neq\xi$$

A comparison of the estimator gains obtained using the operator approach vs the 20-state discrete approximation is shown in Fig. 2. Note that the discretized approach agrees precisely with the operator approach, whereas the discretized approach required the use of a computer to solve a high-order matrix eigenvalue problem.

The real power of the operator approach can be seen in the next example where the sensor accuracy spatially varies. An equally simple example would allow for spatial varying process disturbances.

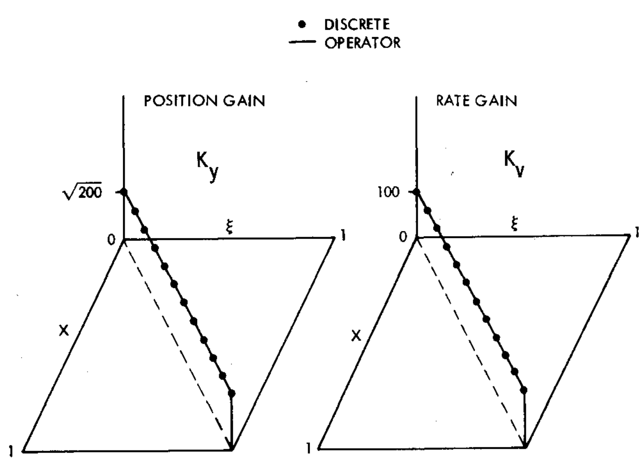

Fig. 2 Comparison of analytic and numerical feedback gains, high gain.

Case II: $Q=10^4$, $R=\sqrt{3}/x$

In this case, no measurement is available at $x=0$ and the accuracy of the measurement increases as x increases. The norm of Q/R is identical to the previous case, so the same expansion of the differential operator applies. Since this norm is still large, local control is expected. In this case, the first-order operator approximations to the optimal estimator gains are

$$K_v=\sqrt{3^{1/2}Qx/R}+\cdots \qquad K_y=2^{1/2}\sqrt[4]{3^{1/2}Qx/R}+\cdots$$

Again, the feedback gains obtained via the operator approach are compared to those obtained via the discrete approximation (see Fig. 3).

By allowing R to be a function of x, it is possible to approximate the case of *spatially discrete* sensors by choosing appropriate continuous approximation functions for $R(x)$.

Case III: $Q=10^{-4}$, $R=1$

For Q/R small, or alternatively, for estimation of the component of the motion due to high-frequency behavior, a different expansion of Eq. (15) gives

$$K_v=-\frac{Q/R}{2D^2}+\frac{(Q/R)^2}{8D^6}+\cdots$$

$$K_y=(Q/R)^{1/2}(-1)^{1/2}/D+\cdots$$

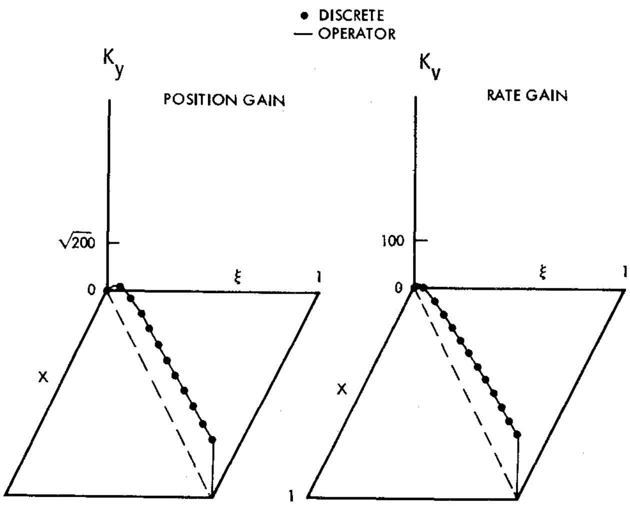

Fig. 3 Comparison of analytic and numerical feedback gains, high gain, spatially varying sensor noise.

As would be expected, for $Q/R=0$, the steady-state estimator gains are zero. For Q/R not equal to zero, the inverse powers of D must be interpreted as inverse operators. For the case of the flexible string, the inverse operator to D^2 that satisfies the boundary conditions is simply the integral operator whose kernel is the Green's function for the string, i.e.,

$$\frac{1}{D^2}f=\int_0^1 g(x,\xi)f(\xi)\,\mathrm{d}\xi$$

where

$$g(x,\xi)=\xi(x-1) \qquad (x>\xi)$$

$$\quad =x(\xi-1) \qquad (x<\xi)$$

Note that $-1/D^2$ has a positive norm as required by the theory. Also, notice that $-1/D^2$ is an integral operator rather than a differential one.

The analytical expression for the $\sqrt{-1/D}$ operator is somewhat more involved. After much work, this expression was found to be

$$g(x,\xi)=\frac{1}{\pi}\log\left|\frac{\sin[\pi(x+\xi)/2]}{\sin[\pi(x-\xi)/2]}\right|$$

In other words,

$$\frac{\sqrt{-1}}{D}f(x)=\int_0^1 g(x,\xi)f(\xi)\,\mathrm{d}\xi$$

For the case of the integral operator feedback operators, the understanding of the feedback term is simpler. In this case, sensor measurements from across the entire structure must be combined with the appropriate weighting to update the state at a given point x. Furthermore, the weighting depends on the specific point x. $K(x,\xi)$ is the term that yields the weighting function at each point in the structure.

There are several very interesting features of this square root operator and how the analytic results compare to the numerical results. First, the analytic results say that an infinite weighting should be applied to the point $x=\xi$. This appears to be a problem at first glance, but upon closer inspection of $K_y(x,\xi)$, it is found that $K_y(x,\xi)$ never appears outside the integral sign and that the singularity in K_y is integrable. Therefore, the expression for K_y retains a meaning. Second, the numerical results yield a finite value for K_y at $x=\xi$, not an infinite value as predicted analytically. It was found, however, that the numerical results yield the *average*

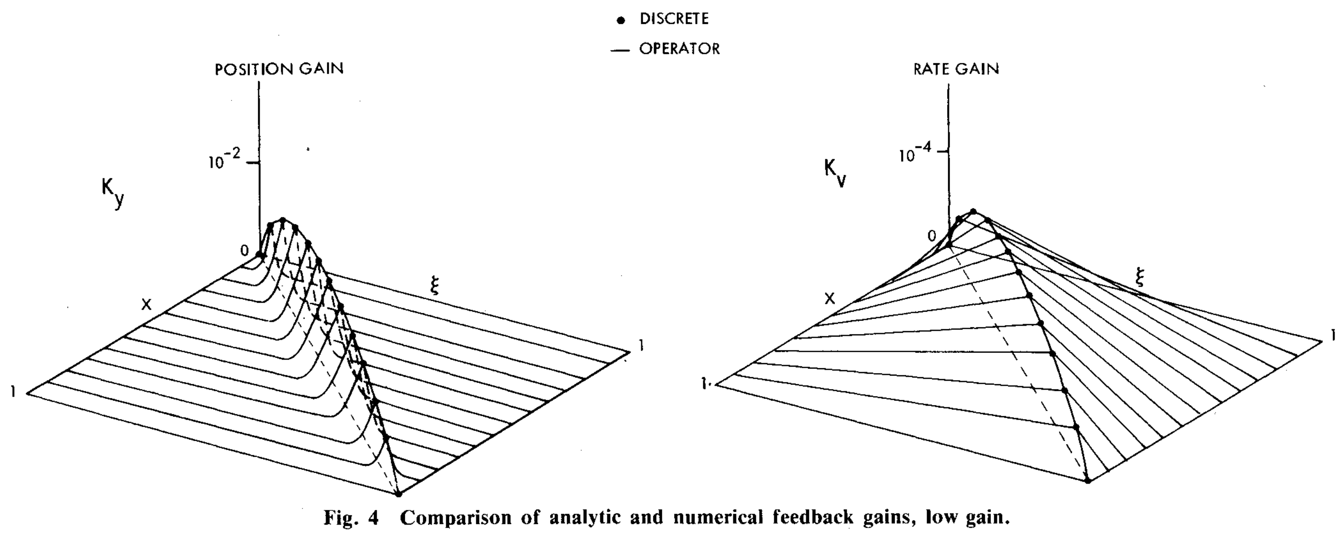

Fig. 4 Comparison of analytic and numerical feedback gains, low gain.

value of K_y over the discretization interval. Again, the numerical results are found to be in agreement with the analytic results. Graphs for both of these operators are shown in Fig. 4.

Conclusions

The use of partial-differential equation (PDE) models for the purpose of estimation and control may eliminate problems associated with model truncation. Based on PDE models, a quadratic performance index, and differential operators, an approach to obtaining the optimal estimator using a partial-differential Ricatti equation has been derived. Using this approach, the closed-loop PDE for the estimated state, and the continuous feedback gains are obtained directly. Using a string in tension, detailed design examples were presented. The results obtained using the differential operator agree very closely in each case with the results obtained using the discretized system model. From these examples, it was observed that two control regimes exist: a low-frequency (spatial or time domain) regime in which local estimation was optimal, and a high-frequency regime in which an operator was required to synthesize the estimator gains. For estimation of the low-frequency modes using a continuous position sensor, local estimation not only guaranteed convergence of the low-frequency estimate error, but of the norm of the full-estimate error. Since often times it will be required to have optimal estimates of the low-frequency modes, and since for the string the truncation of the *operator* did not affect the estimate error stability, it may be possible to base the full estimator on just the local control concept. For the case of spatially discrete measurements, suitable continuous approximation functions may be chosen to represent the spatial measurement noise covariance. This technique may also be extended to include general sensor measurements such as position, slope, curvature, or their time derivatives.

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